Order of Grover's search algorithm with both total and local depolarizing channel errors

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Abstract

Noise is an inherent part of quantum computation. Although there exists a fault-tolerant approach to quantum computing [1], this requires many computational resources. Hence, it is important to analyze how noise affects well-known algorithms. In this paper the effect of noise in Grover's search algorithm [2] is studied. The noise is modeled as both total depolarizing channel (TDCh), and local depolarizing channel (LDCh) in every qubit [3]. An analysis of the order has been made analytically for the TDCh, and bounds have been found for the LDCh.

1. Grover's Search Algorithm

Grover's quantum search algorithm is a quantum search algorithm known to be optimal [4] in finding the *target state* $|t\rangle$ among an unsorted database of $N = 2^n$ elements. The number of *steps* (or query calls) is $k_{Gr} = \left\lfloor \frac{\pi\sqrt{N}}{4} \right\rfloor$.

Grover's search algorithm

1. Set up the superposition state $\rho_0 = |s\rangle\langle s|$, where $|s\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle$.
2. Apply the oracle operator $O = 2\,|t\rangle\,\langle t| - I$
3. Apply the diffusion operator $D = 2\,|s\rangle\,\langle s| - I$.
4. Repeat steps 2 and 3 $\left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor - 1$ times.
5. Perform measurements in the canonical basis in each qubit. The target state will emerge with high probability as $N \gg 1$.

The density matrix obtained after k steps is

$$\rho(k) = |s_k\rangle\langle s_k|,$$

where

$$|s_k\rangle = \sin\left((2k+1)\theta\right)|t\rangle + \cos((2k+1)\theta)|\bar{t}\rangle,$$

$$|\bar{t}\rangle = \frac{1}{\sqrt{N-1}}\sum_{\substack{i=0 \\ i\neq t}}^{N-1}|i\rangle \quad \text{and} \quad \theta = \arcsin\left(\frac{1}{\sqrt{N}}\right).$$

The probability of finding the target state is

$$p(k) = \sin^2\left((2k+1)\theta\right).$$

2. TDCh error model in Grover's algorithm

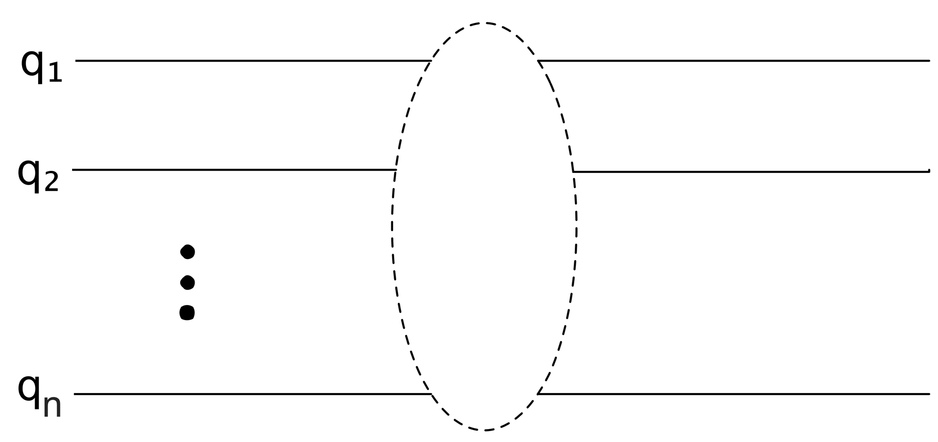

Figure 1: *Total Depolarizing Channel (TDCh) error model.*

We analyze how the error modeled as the Total Depolarizing Channel (TDCh) affects Grover's search algorithm.

- In every step of the algorithm we apply the Grover operator $G = DO$ and the TDCh error

$$\varepsilon(\rho,\gamma) = (1-\gamma)\rho + \gamma\frac{I}{N}.$$

- The probability of finding the marked state is

$$\hat{p}(k,\gamma) = (1-\gamma)^k p(k) + \frac{1-(1-\gamma)^k}{N}.$$

- The maximum probability is found at k_{max}, i.e.

$$k_{max}(\gamma) = \max\left(\left\lfloor \frac{\pi - \arcsin\delta - \arcsin\left(\left[1-\frac{2}{N}\right]\delta\right)}{4\theta} \right\rfloor, 1\right),$$

where $\delta = \sqrt{\frac{1}{1+\left(\frac{4\theta}{\ln(1-\gamma)}\right)^2}}$.

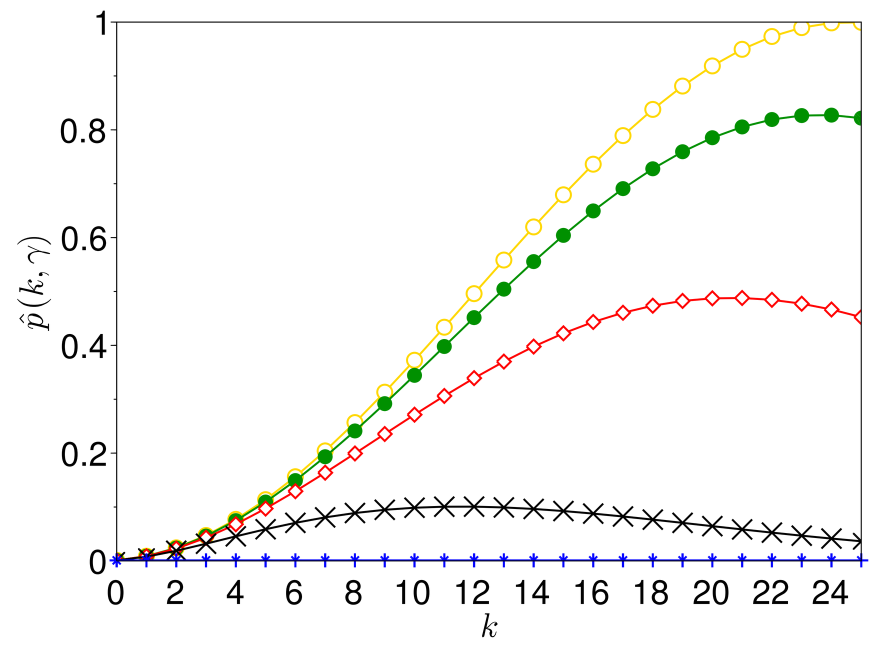

Figure 2: *Probability vs. Number of steps with* $n = 10$ *qubits and different values of* γ, $\gamma = 0$ *(yellow),* $\gamma = \frac{1}{4\sqrt{N}}$ *(green),* $\gamma = \frac{1}{\sqrt{N}}$ *(red),* $\gamma = \frac{4}{\sqrt{N}}$ *(black) and* $\gamma = 1$ *(blue).*

- With k_{max} and $\hat{p}(k,\gamma)$ the estimated order $\left(\frac{k_{max}}{\hat{p}(k_{max},\gamma)}\right)$ becomes

$$\frac{\pi\sqrt{N}}{4}\left(1+\frac{\pi\sqrt{N}}{4}\gamma\left(1-\frac{2}{\pi^2}\right)\right)$$

for $\gamma \ll 1$, and

$$\frac{N}{9-8\gamma}$$

for large values of γ.

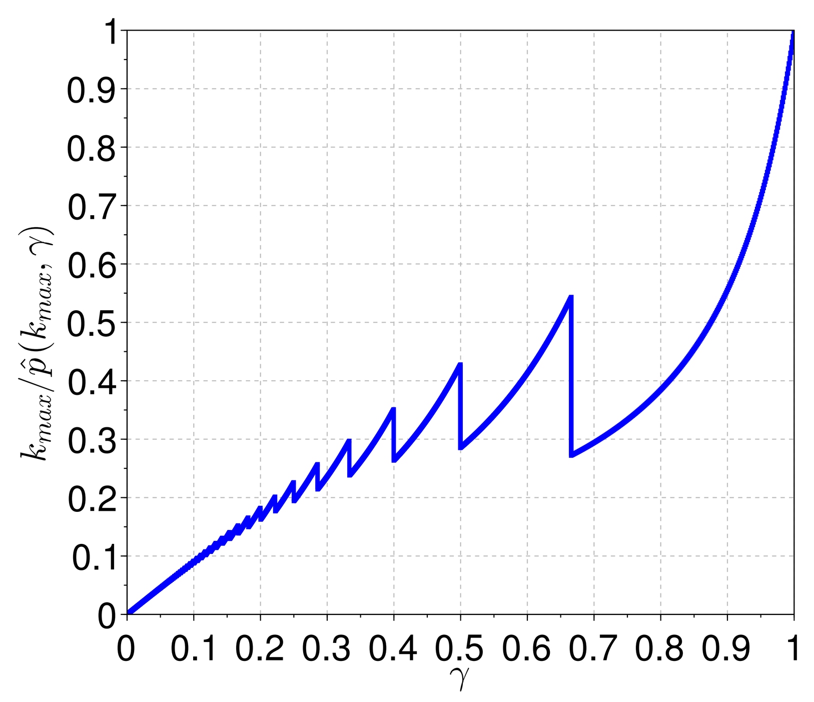

Figure 3: $k_{max}/\hat{p}(k_{max},\gamma)$ *vs.* γ *for* $n = 20$ *qubits.*

3. LDCh error model in Grover's algorithm

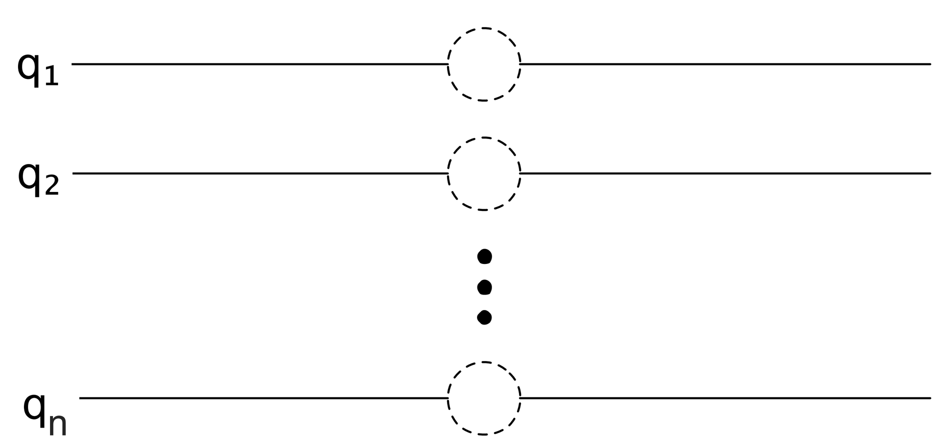

Figure 4: *Local Depolarizing Channel (LDCh) error model.*

Here we analyze the effect of the Local Depolarizing Channel (LDCh) in Grover's search algorithm.

- In every step of the algorithm we apply the Grover operator G and the LDCh error

$$\varepsilon(\rho) = \varepsilon_1(\rho,\alpha)\circ\varepsilon_2(\rho,\alpha)\circ\cdots\circ\varepsilon_n(\rho,\alpha).$$

- The probability of obtaining the marked state after the first step has been determined

$$\hat{p}^L(1,\alpha) = \frac{1}{2^{3n-4}}\left[2^n(2^{n-1}-1)\left(1-\frac{\alpha}{2}\right)^n + \frac{1}{16}(2^n-4)^2\right].$$

- We have obtained an approximation in terms of the TDCh model $\left(\hat{p}(k,\gamma_a(\alpha)) \approx \hat{p}^L(k,\alpha)\right)$:

$$\gamma_a(\alpha) = 1 - \left(1-\frac{\alpha}{2}\right)^n.$$

- We also propose both lower and upper bounds to $\hat{p}^L(k,\alpha)$ in terms of two different γ approximations of the TDCh model:

$$\gamma_l(\alpha) = 1-(1-\alpha)^n \quad \text{and} \quad \gamma_u(\alpha) = \frac{n\alpha}{2+n\alpha},$$

so that the following relationship holds

$$\hat{p}(k,\gamma_l) \le \hat{p}^L(k,\alpha) \le \hat{p}(k,\gamma_u), \quad \forall k \in \mathbb{N}.$$

- Analogously to the TDCh case, the estimated order (stopping at k_{max}) for $\alpha \ll 1$ is

$$\frac{\pi\sqrt{N}}{4} + f\alpha N\log_2 N,$$

where f is a constant in the interval $\left[\frac{\pi^2-4}{32}, \frac{2\pi^2-2}{32}\right]$.

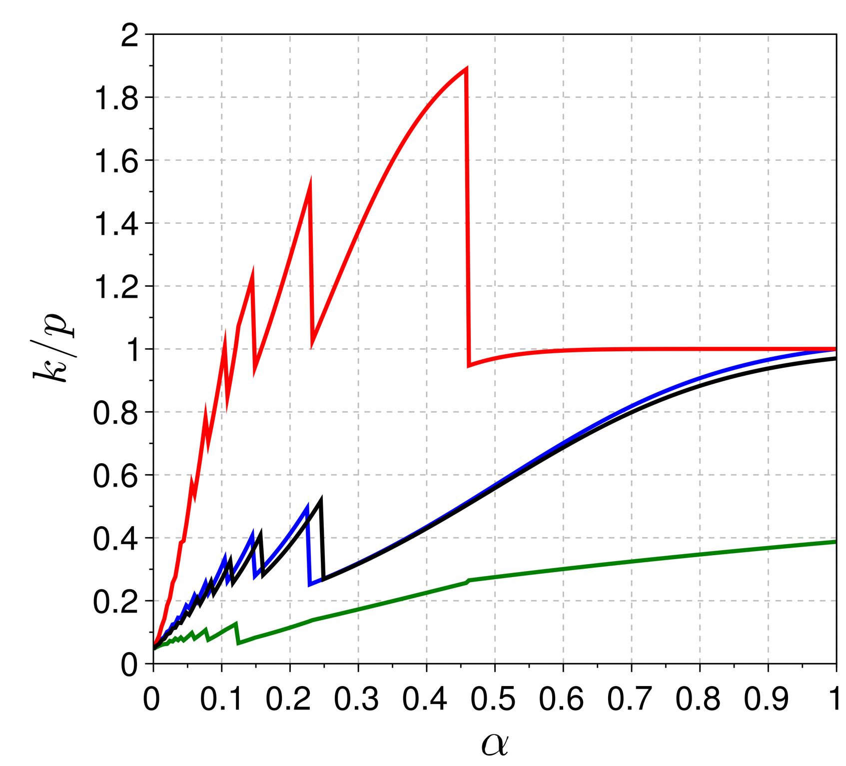

Figure 5: *Bounds and estimated order vs.* α *for* $n = 8$ *qubits.* $\frac{k^L_{max}}{\hat{p}^L(k_{max},\alpha)}$ *(blue), approximation with* $\gamma_a(\alpha)$ *(black) and lower and upper bounds with* $\gamma_l(\alpha)$ *and* $\gamma_u(\alpha)$ *(green and red).*

4. Conclusions

- Total Depolarizing Channel (TDCh)
  - For constant and small values of $\gamma > 0$, the order of the algorithm is $O(N)$, and to mantain the quadratic speedup ($O(\sqrt{N})$) one needs $\gamma \propto \frac{1}{\sqrt{N}}$.
- Local Depolarizing Channel (LDCh)
  - For constant and small values of $\alpha > 0$, the order of the algorithm is $O(N\log_2 N)$, and to mantain the quadratic speedup one needs $\alpha \propto \frac{1}{\sqrt{N}\log_2 N}$.
- Interestingly, local errors affect the algorithm even more than total errors.

References

[1] Daniel Gottesman. *Phys. Rev. A*, **57**:127, 1998.

[2] L.K. Grover. A fast quantum mechanical algorithm for database search. *Proceedings of the 28th Annual ACM Symposium on the Theory of Computing*, 212, 1996.

[3] M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*, Cambridge Univ. Press, 2000.

[4] Christof Zalka. Grover's quantum searching algorithm is optimal. *Phys. Rev. A*, **60**:2746–2751, Oct 1999.

[5] P. Vrana, D. Reeb, D. Reitzner, and M. M. Wolf. *New Journal of Physics*, **16**:073033, 2014.